Typical tangent measures

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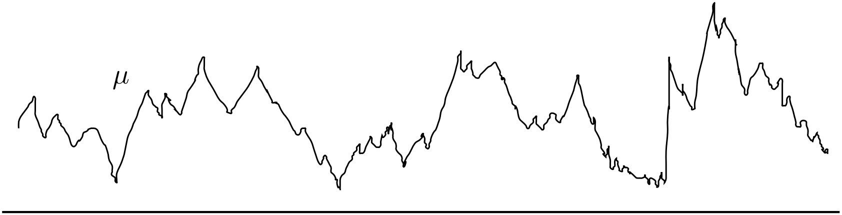
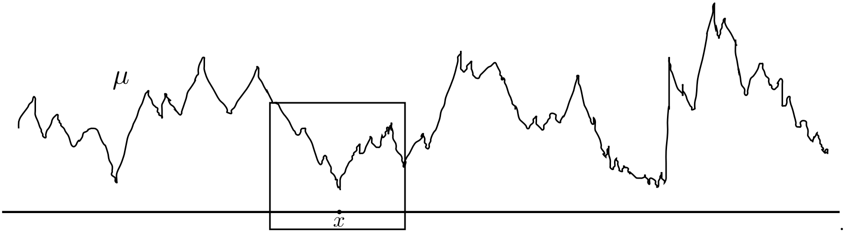
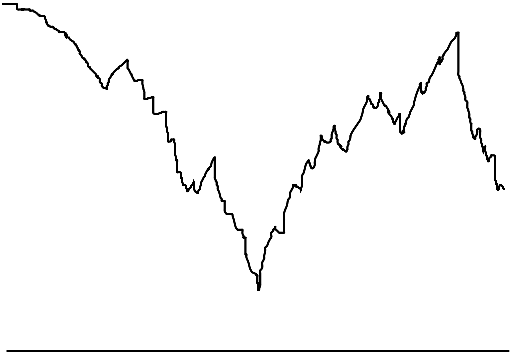
Fix a measure μ on $\mathbb{R}$:



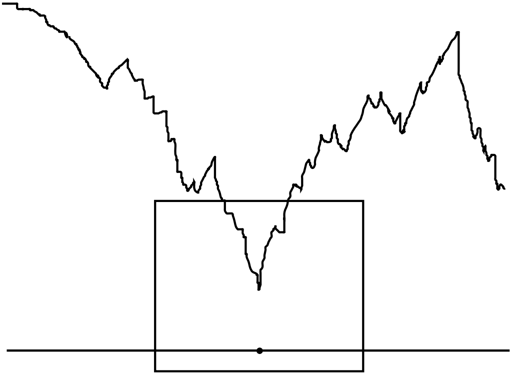
Pick a point $x \in \mathbb{R}$ and take a “photograph” of μ near x:



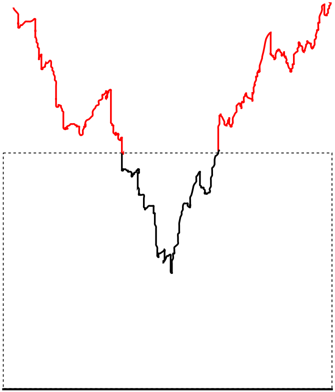
We will see something like this:



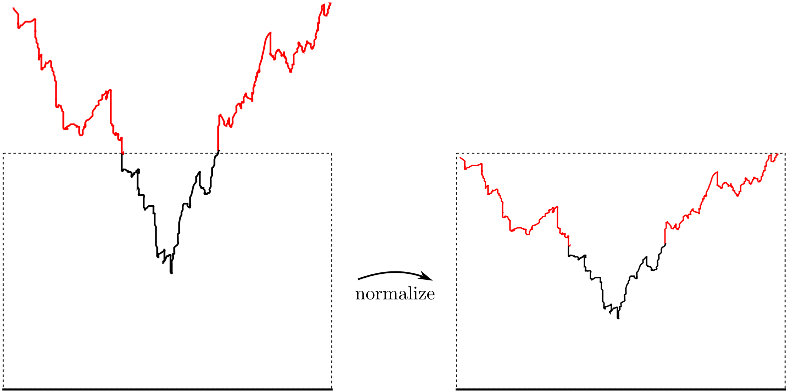
Now take another shot at an even smaller scale:



The photo is now out of the frame:



We have to “photoshop” slightly to fit the picture into the frames:



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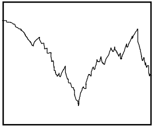

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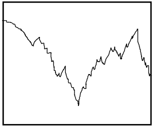

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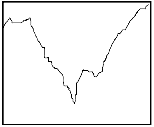

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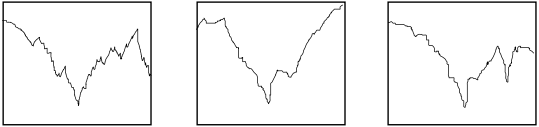

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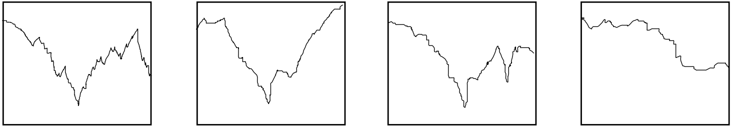

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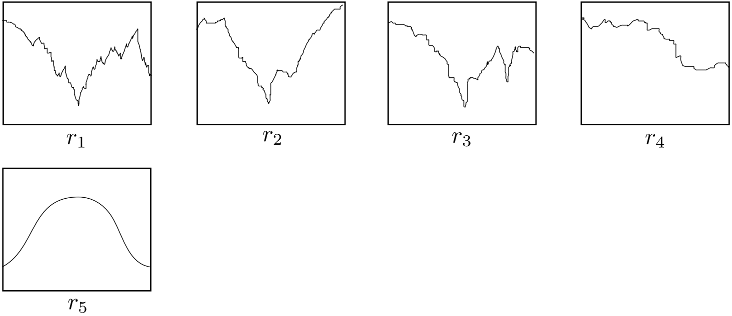

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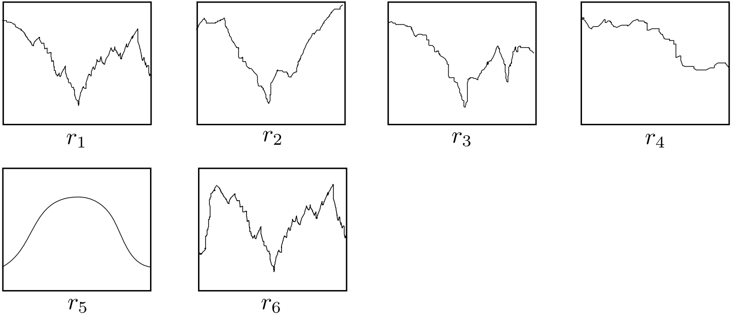

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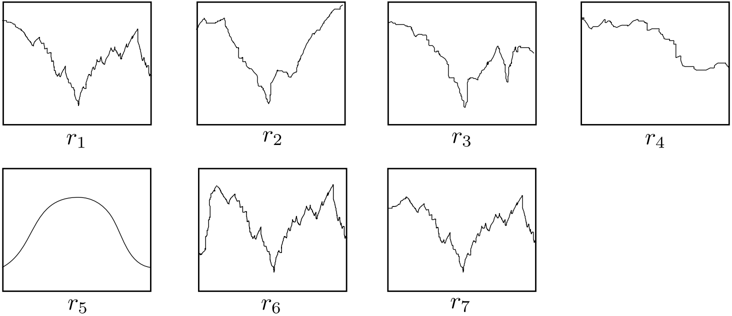

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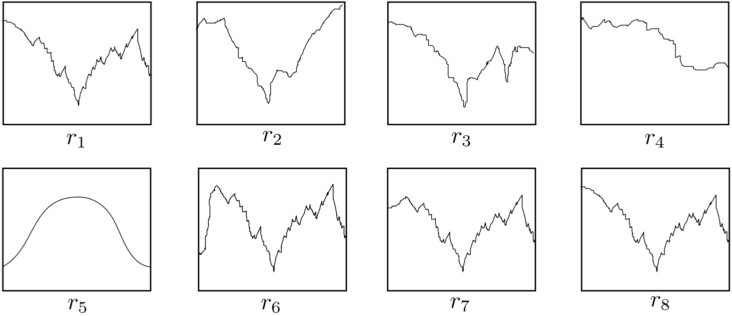

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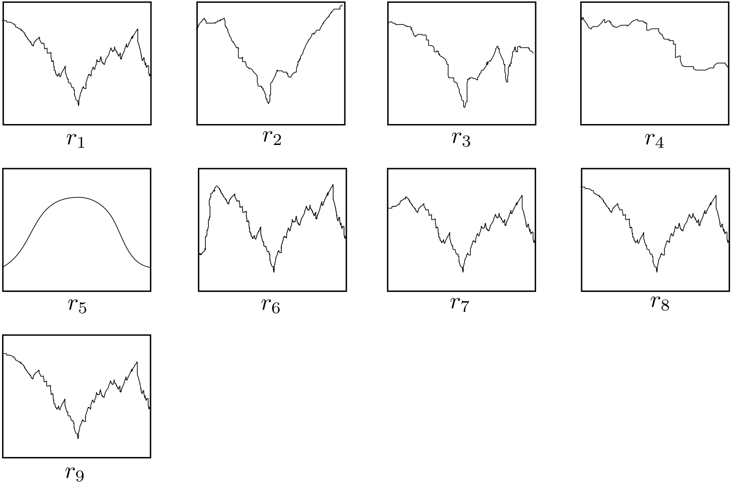

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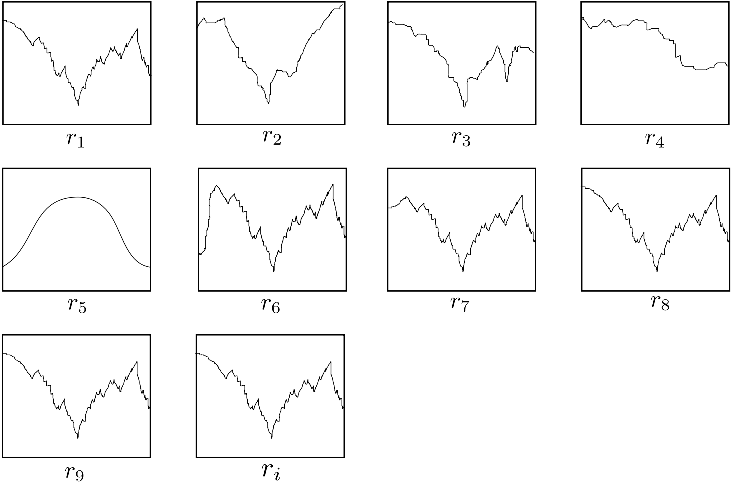

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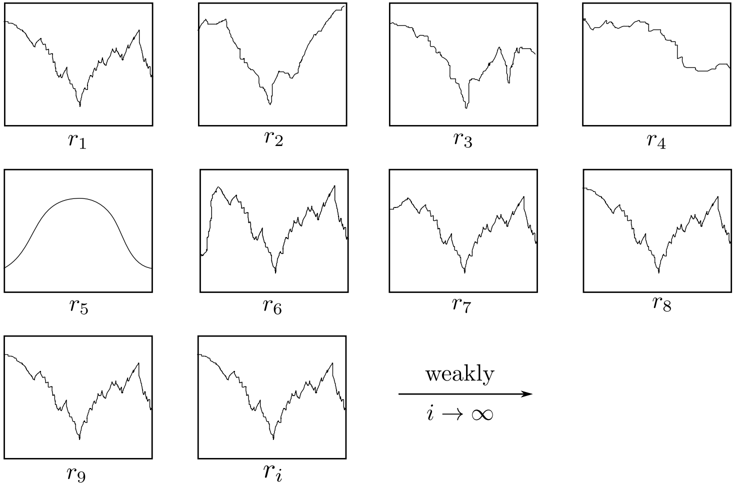

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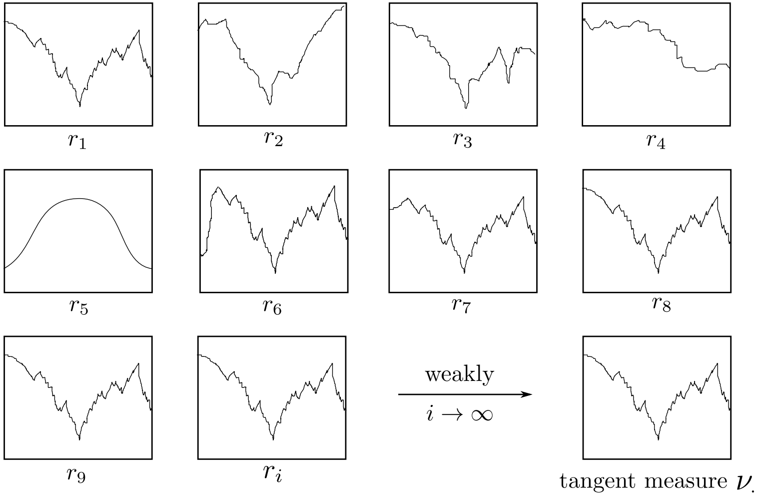

Tangent measures

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Here $T_{x,r} : \mathbb{R}^d \to \mathbb{R}^d$ is the affine homothety that maps the ball $B(x,r)$ onto the unit ball $B(0,1)$.

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Let X be a complete metric space (such as $\mathcal{M}$). A **typical** $x \in X$ satisfies a property P of points $x \in X$ if the set

$$\{x \in X : x \text{ does not satisfy P}\}$$

is **meagre** in X, that is, it is a countable union of sets $E \subset X$ with

$$\operatorname{int} \overline{E} = \varnothing.$$

Tangent measures of typical measures

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The set of tangent measures can be very rich:

Theorem (O'Neil, 1995)

There exists $\mu \in \mathcal{M}$ *such that* $\mathrm{Tan}(\mu, x) = \mathcal{M} \setminus \{0\}$ *at* μ *almost every* $x \in \mathbb{R}^d$*.*

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In fact, such richness is typical:

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Not possible, since:

Proposition

For any $\mu \in \mathcal{M} \setminus \{0\}$ *there exists* $x \in \operatorname{spt} \mu$ *such that either* $\mathcal{L}^d \notin \operatorname{Tan}(\mu, x)$ *or* $\mathcal{L}^d \llcorner [0, \infty)^d \notin \operatorname{Tan}(\mu, x)$.

Here $\mathcal{L}^d$ is the Lebesgue-measure on $\mathbb{R}^d$.

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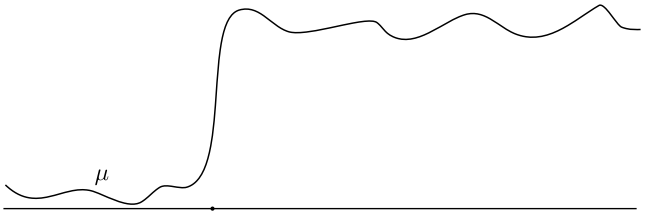
What about case (2)?

Pick a point and scale where μ looks like Lebesgue on $[0, \infty)$.



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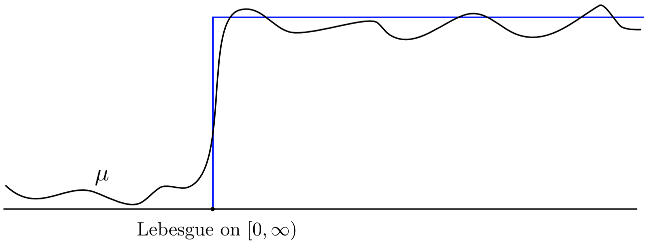
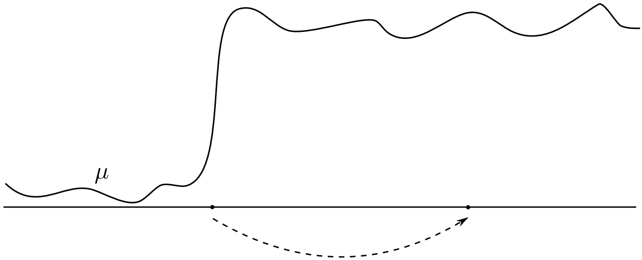
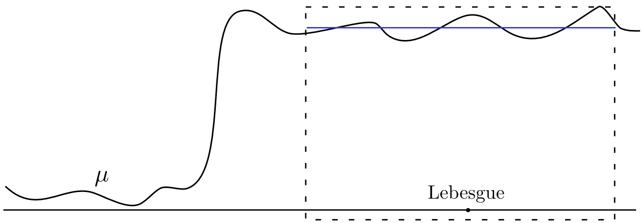


Move a little bit to the right.



Here μ looks like Lebesgue!



Micromeasures

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Let I be any finite set. For $y \in I^n$, $n \in \mathbb{N}$, write

$$[y] = \{x = (x_1, x_2, \dots) \in I^{\mathbb{N}} : x|n := (x_1, ..., x_n) = y\}.$$

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Definition

Let $\mu \in \mathcal{P}$ and $x \in I^{\mathbb{N}}$. A measure $\nu \in \mathcal{P}$ is a **micromeasure** of μ at x if there exists $n_i \nearrow 0$ such that

$$\mu_{x|n_i} \to \nu \quad \text{as } i \to \infty.$$

The set of all such micromeasures is denoted by $\mathrm{micro}(\mu, x)$.

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Theorem

*A typical $\mu \in \mathcal{P}$ satisfies $\mathrm{micro}(\mu, x) = \mathcal{P}$ at **every** $x \in I^{\mathbb{N}}$.*

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Micromeasures are much easier to handle than tangent measures in this situation since

$$\partial[y] = \varnothing \quad \text{for any } y \in I^k, k \in \mathbb{N}.$$

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- Tangent measure/micromeasure *distributions*? (Furstenberg, Zähle, Hochman, Shmerkin...)

References

- T. O'Neil: *A measure with a large set of tangent measures* (1995), Proc. Amer. Math. Soc. **123** (7), 2217 – 2220
- D. Preiss: *Geometry of measures in $\mathbb{R}^d$: distribution, rectifiability, and densities* (1987), Ann. of Math. **125** (3), 537 – 643
- T. Sahlsten: *Tangent measures of typical measures* (2012), preprint at arXiv:1203.4221