Equidistribution modulo 1 and Fourier transforms of equilibrium states

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Joint work with **Thomas Jordan** (Bristol)

Fourier transforms of fractal measures

- **Fourier transform** of a measure μ on $\mathbb{R}$ is

$$\widehat{\mu}(\xi) = \int e^{-2\pi i \xi x} \, d\mu(x), \quad \xi \in \mathbb{R}$$

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- For suitable **Bernoulli convolutions** (Salem), certain **random Cantor measures** (Salem), some random measures arising from **Brownian motion** (Kahane), or specific constructions arising in **Diophantine approximation** (Kaufman) this can be made to happen...

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- ...but for the obvious invariant measures for iterated function systems (with separation conditions) this is not that well studied[1], or does not make any sense...
- ...for example, if μ is **any** probability measure on the $\frac{1}{3}$-Cantor set, then $\widehat{\mu} \not\to 0$!

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$$\widehat{\mu} \to 0 \text{ polynomially.} \qquad \text{(i.e. } |\widehat{\mu}(\xi)| = O(|\xi|^{-\alpha}) \text{ as } |\xi| \to \infty)$$

In fact, this also holds for the s-**conformal measure** on B_N, all **Bernoulli measures** and **Gibbs equilibrium states** on B_N with $\dim \mu > 1/2$, and even on infinite alphabet ($N = \infty$) with a tail assumption on μ.

Some motivation: equidistribution

- $(x_k) \subset \mathbb{R}_+$ **equidistributes** mod 1 if for all intervals I

$$\lim_{N\to\infty} \frac{\sharp\{1 \le k \le N : x_k \bmod 1 \in I\}}{N} = \mathrm{Length}(I)$$

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Theorem (Davenport-Erdős-LeVeque 1963)

Let μ be a measure on $\mathbb{R}$ and $(s_k) \subset \mathbb{N}$. If for any $p \in \mathbb{Z} \setminus \{0\}$ we have

$$\sum_{N=1}^{\infty} \frac{1}{N^3} \sum_{k,n=1}^{N} \widehat{\mu}(p(s_k - s_n)) < \infty,$$

then for μ a.e. x the sequence $(s_k x)_{k \in \mathbb{N}}$ equidistributes modulo 1.

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Corollary

If $\widehat{\mu} \to 0$ polynomially, then for any $(s_k) \subset \mathbb{N}$ strictly increasing sequence the assumption of D-E-L holds.

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Corollary

If $\widehat{\mu} \to 0$ polynomially, then for any $(s_k) \subset \mathbb{N}$ strictly increasing the sequence $(s_k x)_{k \in \mathbb{N}}$ equidistributes mod 1 for μ a.e. x.

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We have this for the Hausdorff- and conformal measures on B_N, and Bernoulli measures and Gibbs equilibrium states on B_N with $\dim > 1/2$ (and on infinite alphabet with a tail assumption).

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"Extends"

Theorem (Hochman-Shmerkin 2013)

If μ is a Gibbs equilibrium state (or just a quasi-Bernoulli measure) on B_N, then μ a.e. x is normal in every base.

Proof?

- Based on the ideas developed by Robert Kaufman on Fourier transform and continued fraction expansions in the 80s[2], where a specific construction of a measure on B_N is given with a polynomially decaying Fourier transform.

[2] Kaufman's result (and an extension by Queffélec and Ramaré later on) led to the solution of Montgomery's conjecture on finding 'N-badly approximable' normal numbers

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- Based on the ideas developed by Robert Kaufman on Fourier transform and continued fraction expansions in the 80s[2], where a specific construction of a measure on B_N is given with a polynomially decaying Fourier transform.
- The key is to exploit the good statistical properties of Gibbs measures (large deviation bounds), which allow us to adapt Kaufman's approach for a 'regular part' of the measure μ

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- "Unfortunately", the proof depends on the link of B_N to continued fraction expansions, which seems to prevent us from extending the result to other iterated function systems beyond the maps

$$x \mapsto \frac{1}{x+a}, \quad a \in \mathbb{N}.$$

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$$x \mapsto \frac{1}{x+a}, \quad a \in \mathbb{N}.$$

- Moreover, if $\dim \mu \leq 1/2$ Kaufman's / Queffélec-Ramaré's strategy collapses for technical reasons...

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References

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