Beyond greedy and lazy beta-shifts

Tuomas Sahlsten

Séminaire Automates, LIAFA
Université Paris Diderot, 24.5.2013

Joint work with **Bing Li** and **Tony Samuel**

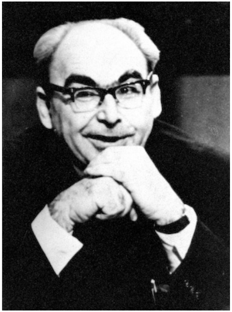
UNIVERSITY of OULU
OULUN YLIOPISTO

University of BRISTOL

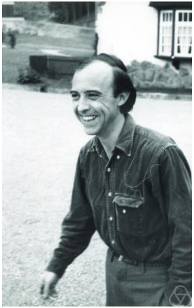
Universität Bremen

My co-authors

- **A. Rényi**: *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hung. 8 (1957), 477–493;
- **W. Parry**: *On the β-expansions of real numbers*, Acta Math. Acad. Sci. Hung. 11 (1960), 401–416.



Alfréd Rényi
20.3.1921 – 1.2.1970



William (Bill) Parry
3.7.1934 – 30.8.2006

Base-2 expansions

$$x = \sum_{i=1}^{\infty} w_i 2^{-i}, \quad w_i \in \{0, 1\}$$

Base-2 expansions

$$x = \sum_{i=1}^{\infty} w_i 2^{-i}, \quad w_i \in \{0, 1\}$$

Base-β expansions

$$x = \sum_{i=1}^{\infty} w_i \beta^{-i}, \quad w_i \in \{0, 1\}$$

$$1 < \beta \leq 2$$

Base-β expansions

$$x = \sum_{i=1}^{\infty} w_i \beta^{-i}, \quad w_i \in \{0, 1\}$$

$$1 < \beta \leq 2$$

Theorem (Sidorov 2003)

If $1 < \beta < 2$, then Lebesgue a.e. $x \in [0, \frac{1}{\beta - 1}]$ has continuum β-expansions.

Theorem (Sidorov 2003)

If $1 < \beta < 2$, then Lebesgue a.e. $x \in [0, \frac{1}{\beta-1}]$ has continuum β-expansions.

Theorem (Sidorov 2003)

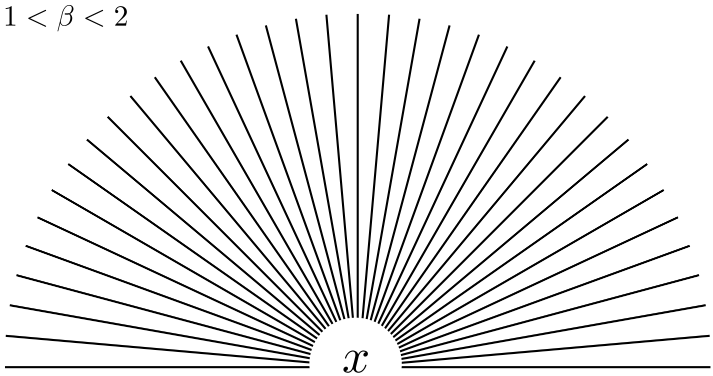
If $1 < \beta < 2$, then Lebesgue a.e. $x \in [0, \frac{1}{\beta - 1}]$ has continuum β-expansions.

$\beta = 2$

$w'_1 w'_2 w'_3 \ldots$ $\qquad x \qquad$ $w_1 w_2 w_3 \ldots$

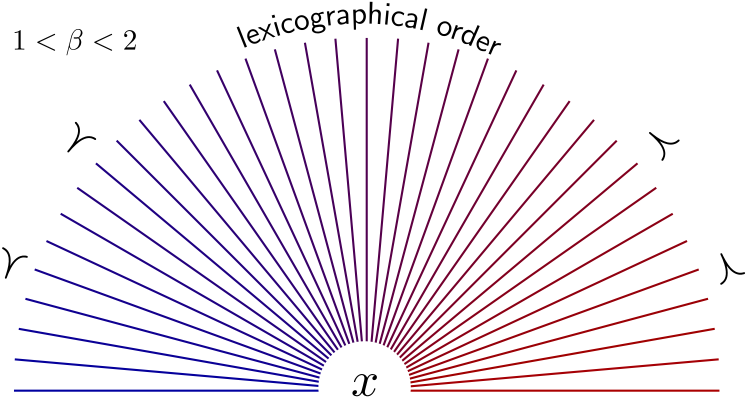
Theorem (Sidorov 2003)

If $1 < \beta < 2$, then Lebesgue a.e. $x \in [0, \frac{1}{\beta-1}]$ has continuum β-expansions.



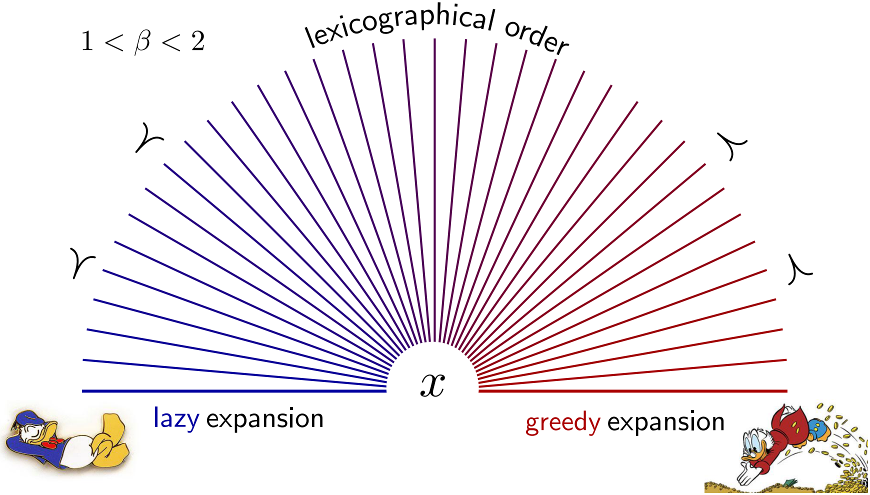
Theorem (Sidorov 2003)

If $1 < \beta < 2$, then Lebesgue a.e. $x \in [0, \frac{1}{\beta-1}]$ has continuum β-expansions.



Theorem (Sidorov 2003)

If $1 < \beta < 2$, then Lebesgue a.e. $x \in [0, \frac{1}{\beta-1}]$ has continuum β-expansions.

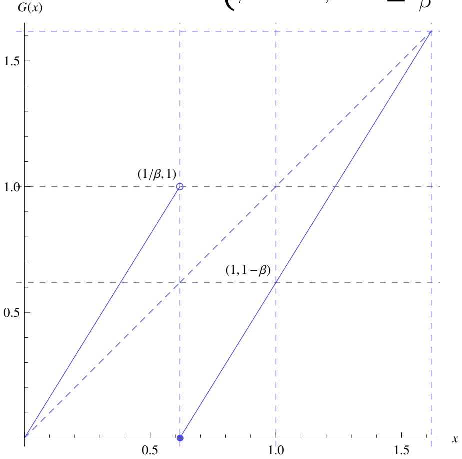


Generating greedy digits?

Generating greedy digits?

- Greedy β-transformation $G : [0, \frac{1}{\beta-1}] \to [0, \frac{1}{\beta-1}]$,

$$G(x) = \begin{cases} \beta x, & x < \frac{1}{\beta}; \\ \beta x - 1, & x \geq \frac{1}{\beta}. \end{cases}$$



Generating greedy digits?

$$\boxed{G(x) = \begin{cases} \beta x, & x < \frac{1}{\beta} \\ \beta x - 1, & x \geq \frac{1}{\beta} \end{cases}}$$

Let $G^0(x) = x$ and $G^i = \underbrace{G \circ G \circ \cdots \circ G}_{i \text{ times}}$

Generating greedy digits?

$$G(x) = \begin{cases} \beta x, & x < \frac{1}{\beta} \\ \beta x - 1, & x \geq \frac{1}{\beta} \end{cases}$$

Let $G^0(x) = x$ and $G^i = \underbrace{G \circ G \circ \cdots \circ G}_{i \text{ times}}$

Then for

$$w_i(x) := \lfloor \beta G^{i-1}(x) \rfloor \in \{0, 1\}$$

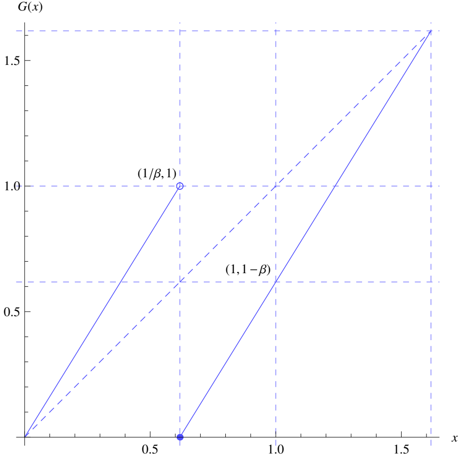
the sequence $w_1(x)w_2(x) \cdots \in \{0,1\}^{\mathbb{N}}$ is the greedy expansion of x...

...thus we obtain the greedy expansion of x from the **orbit** $\{G^i(x)\}_{i \in \mathbb{N}}$ under the greedy β-transformation!

Trapping region

Trapping region

Region $[0, 1]$ attracts and **traps** every orbit $G^i(x)$ for $x < \frac{1}{\beta-1}$...



... so the greedy expansions of points in $[0, \frac{1}{\beta-1})$ can be determined from the points in $[0, 1]$.

Generating **other** digits?

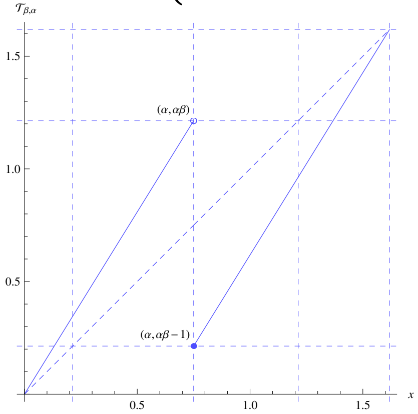
Generating **other** digits?

- Fix $\alpha \in [\frac{1}{\beta}, \frac{1}{\beta(\beta-1)}]$.

Generating **other** digits?

- Fix $\alpha \in [\frac{1}{\beta}, \frac{1}{\beta(\beta-1)}]$. Let $\mathcal{T}_{\beta,\alpha} : [0, \frac{1}{\beta-1}] \to [0, \frac{1}{\beta-1}]$ be the map

$$\mathcal{T}_{\beta,\alpha}(x) = \begin{cases} \beta x, & x < \alpha; \\ \beta x - 1, & x \geq \alpha. \end{cases}$$



- The trapping region for orbits $\{\mathcal{T}^i_{\beta,\alpha}(x)\}_{i\in\mathbb{N}}$ is $[\alpha\beta - 1, \alpha\beta]$
- Then $\mathcal{T}_{\beta,\frac{1}{\beta}} = G$ and $\mathcal{T}_{\beta,\frac{1}{\beta(\beta-1)}} = L$, the lazy β-transformation.

Scale $\mathcal{T}_{\beta,\alpha}$ from $[0, \frac{1}{\beta-1}]$ to $[0, 1]$

$$\boxed{\mathcal{T}_{\beta,\alpha}(x) = \begin{cases} \beta x, & x < \alpha; \\ \beta x - 1, & x \geq \alpha. \end{cases}}$$

- Scale everything by $\beta - 1$, let $\boxed{p := \alpha(\beta - 1) \in [1 - \frac{1}{\beta}, \frac{1}{\beta}]}$,

Scale $\mathcal{T}_{\beta,\alpha}$ from $[0, \frac{1}{\beta-1}]$ to $[0,1]$

$$\boxed{\mathcal{T}_{\beta,\alpha}(x) = \begin{cases} \beta x, & x < \alpha; \\ \beta x - 1, & x \geq \alpha. \end{cases}}$$

- Scale everything by $\beta - 1$, let $\boxed{p := \alpha(\beta - 1) \in [1 - \frac{1}{\beta}, \frac{1}{\beta}]}$, define $T^{\pm}_{\beta,p} : [0,1] \circlearrowleft$ by

$$T^{+}_{\beta,p}(x) = \begin{cases} \beta x, & x < p; \\ \beta x + 1 - \beta, & x \geq p. \end{cases}$$

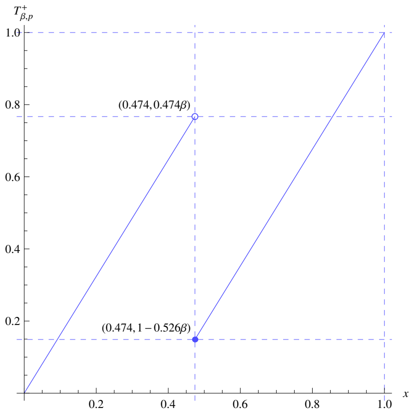
and

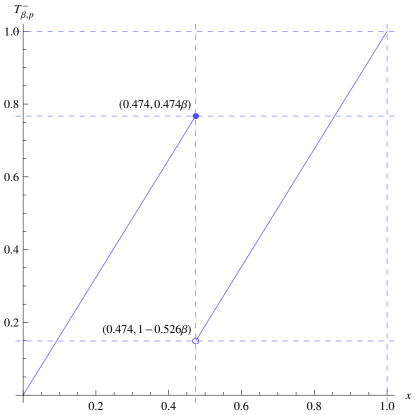
$$T^{-}_{\beta,p}(x) = \begin{cases} \beta x, & x \leq p; \\ \beta x + 1 - \beta, & x > p. \end{cases}$$

Scale $\mathcal{T}_{\beta,\alpha}$ from $[0, \frac{1}{\beta-1}]$ to $[0, 1]$

$$T^+_{\beta,p}(x) = \begin{cases} \beta x, & x < p \\ \beta x + 1 - \beta, & x \geq p \end{cases}$$

$$T^-_{\beta,p}(x) = \begin{cases} \beta x, & x \leq p \\ \beta x + 1 - \beta, & x > p \end{cases}$$





(β, p)-shifts

(β, p)-shifts

- The i:th digit of the $(\beta, p, \pm)$-**expansion** $\tau^{\pm}_{\beta,p}(x)$ of $x \in [0,1]$ is

$$[\tau^{\pm}_{\beta,p}(x)]_i := \begin{cases} 0 & \text{if } (T^{+}_{\beta,p})^{i-1}(x) \leq p \text{ or resp. } (T^{-}_{\beta,p})^{i-1}(x) < p, \\ 1 & \text{otherwise.} \end{cases}$$

(β, p)-shifts

- The i:th digit of the $(\beta, p, \pm)$-**expansion** $\tau^{\pm}_{\beta,p}(x)$ of $x \in [0,1]$ is

$$[\tau^{\pm}_{\beta,p}(x)]_i := \begin{cases} 0 & \text{if } (T^{+}_{\beta,p})^{i-1}(x) \leq p \text{ or resp. } (T^{-}_{\beta,p})^{i-1}(x) < p, \\ 1 & \text{otherwise.} \end{cases}$$

- These digits give an actual β-expansion of the point $\frac{x}{\beta-1} \in [0, \frac{1}{\beta-1}]$.

(β, p)-shifts

- The i:th digit of the $(\beta, p, \pm)$-**expansion** $\tau^{\pm}_{\beta,p}(x)$ of $x \in [0, 1]$ is

$$[\tau^{\pm}_{\beta,p}(x)]_i := \begin{cases} 0 & \text{if } (T^{+}_{\beta,p})^{i-1}(x) \le p \text{ or resp. } (T^{-}_{\beta,p})^{i-1}(x) < p, \\ 1 & \text{otherwise.} \end{cases}$$

- These digits give an actual β-expansion of the point $\frac{x}{\beta-1} \in [0, \frac{1}{\beta-1}]$.
- Write

$$\Omega^{\pm}_{\beta,p} = \{w \in \{0,1\}^{\mathbb{N}} : w \text{ is a } (\beta, p, \pm)\text{-expansion for some } x \in [0, 1]\}$$

(β, p)-shifts

- The i:th digit of the $(\beta, p, \pm)$-**expansion** $\tau^{\pm}_{\beta,p}(x)$ of $x \in [0,1]$ is

$$[\tau^{\pm}_{\beta,p}(x)]_i := \begin{cases} 0 & \text{if } (T^{+}_{\beta,p})^{i-1}(x) \leq p \text{ or resp. } (T^{-}_{\beta,p})^{i-1}(x) < p, \\ 1 & \text{otherwise.} \end{cases}$$

- These digits give an actual β-expansion of the point $\frac{x}{\beta-1} \in [0, \frac{1}{\beta-1}]$.
- Write

$$\Omega^{\pm}_{\beta,p} = \{w \in \{0,1\}^{\mathbb{N}} : w \text{ is a } (\beta,p,\pm)\text{-expansion for some } x \in [0,1]\}$$

- Then

$$\overline{\Omega}^{+}_{\beta,p} = \overline{\Omega}^{-}_{\beta,p} = \Omega^{+}_{\beta,p} \cup \Omega^{-}_{\beta,p} =: \Omega_{\beta,p}$$

  and the (β, p)-shift is $\sigma_{\beta,p} = \sigma|_{\Omega_{\beta,p}}$.

(β, p)-shifts

- The i:th digit of the $(\beta, p, \pm)$-**expansion** $\tau^{\pm}_{\beta,p}(x)$ of $x \in [0,1]$ is

$$[\tau^{\pm}_{\beta,p}(x)]_i := \begin{cases} 0 & \text{if } (T^{+}_{\beta,p})^{i-1}(x) \leq p \text{ or resp. } (T^{-}_{\beta,p})^{i-1}(x) < p, \\ 1 & \text{otherwise.} \end{cases}$$

- These digits give an actual β-expansion of the point $\frac{x}{\beta-1} \in [0, \frac{1}{\beta-1}]$.
- Write

$$\Omega^{\pm}_{\beta,p} = \{w \in \{0,1\}^{\mathbb{N}} : w \text{ is a } (\beta,p,\pm)\text{-expansion for some } x \in [0,1]\}$$

- Then

$$\overline{\Omega}^{+}_{\beta,p} = \overline{\Omega}^{-}_{\beta,p} = \Omega^{+}_{\beta,p} \cup \Omega^{-}_{\beta,p} =: \Omega_{\beta,p}$$

and the (β, p)-shift is $\sigma_{\beta,p} = \sigma|_{\Omega_{\beta,p}}$.

Theorem (Hubbard, Sparrow 1990)

$$\Omega^{+}_{\beta,p} = \left\{w \in \{0,1\}^{\mathbb{N}} : \sigma^n(w) \prec \tau^{-}_{\beta,p}(p) \text{ or } \sigma^n(w) \succeq \tau^{+}_{\beta,p}(p), \text{ for all } n \in \mathbb{N}\right\}$$

$$\Omega^{-}_{\beta,p} = \left\{w \in \{0,1\}^{\mathbb{N}} : \sigma^n(w) \preceq \tau^{-}_{\beta,p}(p) \text{ or } \sigma^n(w) \succ \tau^{+}_{\beta,p}(p), \text{ for all } n \in \mathbb{N}\right\}$$

Dynamics of (β, p)-shifts

Dynamics of (β, p)-shifts

- A subshift $(S, \sigma|_S)$ is a **subshift of finite type** (SFT), if it can be specfied by finitely many forbidden words.

Dynamics of (β, p)-shifts

- A subshift $(S, \sigma|_S)$ is a **subshift of finite type** (SFT), if it can be specfied by finitely many forbidden words.

Theorem (Parry 1960)

The greedy β-shift $(\Omega_{\beta,\frac{1}{\beta}}, \sigma_{\beta,\frac{1}{\beta}})$ is a SFT if and only if the orbit $\{G^i(1)\}_{i\in\mathbb{N}}$ is periodic

Dynamics of (β, p)-shifts

- A subshift $(S, \sigma|_S)$ is a **subshift of finite type** (SFT), if it can be specfied by finitely many forbidden words.

Theorem (Parry 1960)

The greedy β-shift $(\Omega_{\beta,\frac{1}{\beta}}, \sigma_{\beta,\frac{1}{\beta}})$ is a SFT if and only if the orbit $\{G^i(1)\}_{i\in\mathbb{N}}$ is periodic

Theorem

The (β, p)-shift $(\Omega_{\beta,p}, \sigma_{\beta,p})$ is a SFT if and only if both the orbits $\{(T^-_{\beta,p})^i(p)\}_{i\in\mathbb{N}}$ and $\{(T^+_{\beta,p})^i(p)\}_{i\in\mathbb{N}}$ are periodic

- For $\beta > 2$ an analogous result was proven by Wilkinson (1975)

Idea of the proof I

$\boxed{\Longrightarrow}$ Fix $1 < \beta < 2$ and $1 - \frac{1}{\beta} < p < \frac{1}{\beta}$.

Idea of the proof I

$\boxed{\Longrightarrow}$ Fix $1 < \beta < 2$ and $1 - \frac{1}{\beta} < p < \frac{1}{\beta}$.

- For $w \in \Omega^{\pm}_{\beta,p}$, define branching numbers:

$$N_0(w) := \left| \left\{ k > 2 : w_1 w_2 \dots w_{k-1} 1 \in \Omega^{-}_{\beta,p}|_k \text{ and } w_k = 0 \right\} \right|$$

$$N_1(w) := \left| \left\{ k > 2 : w_1 w_2 \dots w_{k-1} 0 \in \Omega^{+}_{\beta,p}|_k \text{ and } w_k = 1 \right\} \right|$$

Idea of the proof I

$\boxed{\Longrightarrow}$ Fix $1 < \beta < 2$ and $1 - \frac{1}{\beta} < p < \frac{1}{\beta}$.

- For $w \in \Omega^{\pm}_{\beta,p}$, define branching numbers:

$$N_0(w) := \left|\left\{k > 2 : w_1 w_2 \dots w_{k-1} 1 \in \Omega^{-}_{\beta,p}|_k \text{ and } w_k = 0\right\}\right|$$

$$N_1(w) := \left|\left\{k > 2 : w_1 w_2 \dots w_{k-1} 0 \in \Omega^{+}_{\beta,p}|_k \text{ and } w_k = 1\right\}\right|$$

Lemma

- If $\{(T^{-}_{\beta,p})^i(p)\}_{i \in \mathbb{N}}$ is not periodic, then $N_0(\sigma(\tau^{-}_{\beta,p}(p))) = +\infty$
- If $\{(T^{+}_{\beta,p})^i(p)\}_{i \in \mathbb{N}}$ is not periodic, then $N_1(\sigma(\tau^{+}_{\beta,p}(p))) = +\infty$

Idea of the proof II

$$\boxed{N_0(w) = \left|\left\{k > 2 : w_1 w_2 \dots w_{k-1} 1 \in \Omega^-_{\beta,p}|_k \text{ and } w_k = 0\right\}\right|}$$

Idea of the proof II

$$N_0(w) = \left|\left\{k > 2 : w_1 w_2 \dots w_{k-1}1 \in \Omega^-_{\beta,p}|_k \text{ and } w_k = 0\right\}\right|$$

- Suppose the orbit $\{(T^-_{\beta,p})^i(p)\}_{i\in\mathbb{N}}$ is not periodic...

Idea of the proof II

$$N_0(w) = \left|\left\{k > 2 : w_1 w_2 \dots w_{k-1} 1 \in \Omega^-_{\beta,p}|_k \text{ and } w_k = 0\right\}\right|$$

- Suppose the orbit $\{(T^-_{\beta,p})^i(p)\}_{i\in\mathbb{N}}$ is not periodic...
- ...then $N_0(w) = \infty$ for $w = w_1 w_2 w_3 \cdots := \sigma(\tau^-_{\beta,p}(p))$.

Idea of the proof II

$$N_0(w) = \left|\left\{k > 2 : w_1 w_2 \dots w_{k-1} 1 \in \Omega^-_{\beta,p}|_k \text{ and } w_k = 0\right\}\right|$$

- Suppose the orbit $\{(T^-_{\beta,p})^i(p)\}_{i\in\mathbb{N}}$ is not periodic...
- ...then $N_0(w) = \infty$ for $w = w_1 w_2 w_3 \cdots := \sigma(\tau^-_{\beta,p}(p))$.
- Thus for infinitely many $k \in \mathbb{N}$

$$w_1 w_2 \dots w_{k-1} 1 \in \Omega_{\beta,p}|_k \quad \text{and} \quad w_k = 0$$

Idea of the proof II

$$N_0(w) = \left|\left\{k > 2 : w_1 w_2 \dots w_{k-1} 1 \in \Omega^-_{\beta,p}|_k \text{ and } w_k = 0\right\}\right|$$

- Suppose the orbit $\{(T^-_{\beta,p})^i(p)\}_{i\in\mathbb{N}}$ is not periodic...
- ...then $N_0(w) = \infty$ for $w = w_1 w_2 w_3 \cdots := \sigma(\tau^-_{\beta,p}(p))$.
- Thus for infinitely many $k \in \mathbb{N}$

$$w_1 w_2 \dots w_{k-1} 1 \in \Omega_{\beta,p}|_k \quad \text{and} \quad w_k = 0$$

- ...but

$$0 w_1 w_2 \dots w_{k-1} 1 \notin \Omega_{\beta,p}|_{k+1}$$

since

$$0 w_1 w_2 \dots w_{k-1} 0 = \tau^-_{\beta,p}(p)|_{k+1}$$

(the word $\tau^-_{\beta,p}(p)$ always begins with the subword 01)

Idea of the proof II

$$N_0(w) = \left|\left\{k > 2 : w_1 w_2 \dots w_{k-1} 1 \in \Omega^-_{\beta,p}|_k \text{ and } w_k = 0\right\}\right|$$

- Suppose the orbit $\{(T^-_{\beta,p})^i(p)\}_{i\in\mathbb{N}}$ is not periodic...
- ...then $N_0(w) = \infty$ for $w = w_1 w_2 w_3 \cdots := \sigma(\tau^-_{\beta,p}(p))$.
- Thus for infinitely many $k \in \mathbb{N}$

$$w_1 w_2 \dots w_{k-1} 1 \in \Omega_{\beta,p}|_k \quad \text{and} \quad w_k = 0$$

- ...but

$$0 w_1 w_2 \dots w_{k-1} 1 \notin \Omega_{\beta,p}|_{k+1}$$

  since

$$0 w_1 w_2 \dots w_{k-1} 0 = \tau^-_{\beta,p}(p)|_{k+1}$$

  (the word $\tau^-_{\beta,p}(p)$ always begins with the subword 01)
- Thus $(\Omega_{\beta,p}, \sigma_{\beta,p})$ is not SFT.

Idea of the proof II

$$\boxed{N_0(w) = \left|\left\{k > 2 : w_1 w_2 \dots w_{k-1} 1 \in \Omega^-_{\beta,p}|_k \text{ and } w_k = 0\right\}\right|}$$

- Suppose the orbit $\{(T^-_{\beta,p})^i(p)\}_{i\in\mathbb{N}}$ is not periodic...
- ...then $N_0(w) = \infty$ for $w = w_1 w_2 w_3 \cdots := \sigma(\tau^-_{\beta,p}(p))$.
- Thus for infinitely many $k \in \mathbb{N}$

$$w_1 w_2 \dots w_{k-1} 1 \in \Omega_{\beta,p}|_k \quad \text{and} \quad w_k = 0$$

- ...but

$$0 w_1 w_2 \dots w_{k-1} 1 \notin \Omega_{\beta,p}|_{k+1}$$

  since

$$0 w_1 w_2 \dots w_{k-1} 0 = \tau^-_{\beta,p}(p)|_{k+1}$$

  (the word $\tau^-_{\beta,p}(p)$ always begins with the subword 01)
- Thus $(\Omega_{\beta,p}, \sigma_{\beta,p})$ is not SFT.

For the direction $\boxed{\Longrightarrow}$ if the orbits $\{(T^\pm_{\beta,p})^i(p)\}_{i\in\mathbb{N}}$ are $M^\pm$ periodic, $M^\pm \in \mathbb{N}$, then $(\Omega_{\beta,p}, \sigma_{\beta,p})$ is SFT with step $M := M^- M^+$.

Where are SFTs?

Where are SFTs?

- Define the **parameter space**

$$\Delta = \{(\beta, p) \in (1,2) \times (0,1) : p \in [1 - \tfrac{1}{\beta}, \tfrac{1}{\beta}]\}$$

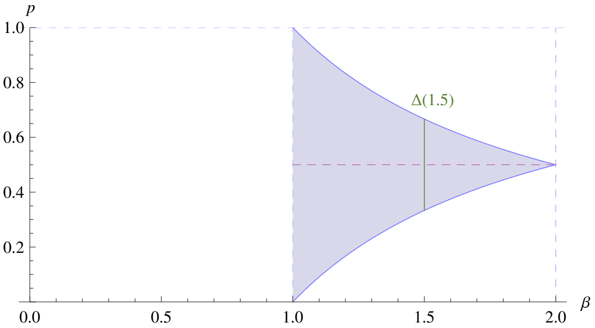
Where are SFTs?

- Define the **parameter space**

$$\Delta = \{(\beta, p) \in (1,2) \times (0,1) : p \in [1 - \tfrac{1}{\beta}, \tfrac{1}{\beta}]\}$$

and vertical **fibres**

$$\Delta(\beta) = \{\beta\} \times [1 - \tfrac{1}{\beta}, \tfrac{1}{\beta}]$$



Where are SFTs?

Theorem (Parry 1960)

The collection $\{\beta \in (1,2)$: greedy β-shift is SFT$\}$ is dense in $(1,2)$

Where are SFTs?

Theorem (Parry 1960)

The collection $\{\beta \in (1,2) : \text{greedy } \beta\text{-shift is SFT}\}$ is dense in $(1,2)$

Conjecture

The collection $\Delta_{\mathrm{SFT}} := \{(\beta, p) \in \Delta : (\beta, p)\text{-shift is SFT}\}$ is dense in Δ

Where are SFTs?

Theorem (Parry 1960)

The collection $\{\beta \in (1,2) : \text{greedy } \beta\text{-shift is SFT}\}$ is dense in $(1,2)$

Conjecture

The collection $\Delta_{\mathrm{SFT}} := \{(\beta, p) \in \Delta : (\beta, p)\text{-shift is SFT}\}$ is dense in Δ

- An subshift $(S, \sigma|_S)$ is **transitive**, if there exists $w \in S$ s.t. the orbit $\{\sigma^i(w)\}_{i \in \mathbb{N}}$ is dense in S.

Where are SFTs?

Theorem (Parry 1960)

The collection $\{\beta \in (1,2) :$ greedy β-shift is SFT$\}$ is dense in $(1,2)$

Conjecture

The collection $\Delta_{\mathrm{SFT}} := \{(\beta, p) \in \Delta : (\beta, p)$-shift is SFT$\}$ is dense in Δ

- An subshift $(S, \sigma|_S)$ is **transitive**, if there exists $w \in S$ s.t. the orbit $\{\sigma^i(w)\}_{i \in \mathbb{N}}$ is dense in S.

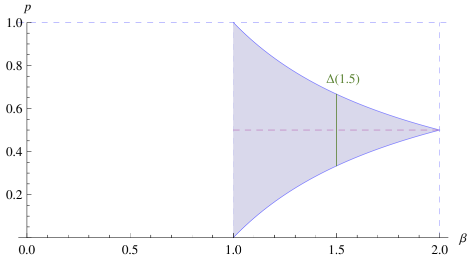
Theorem (Palmer 1979)

The set $\{(\beta, p) \in \Delta : (\beta, p)$-shift is transitive$\}$ is **not** dense in Δ

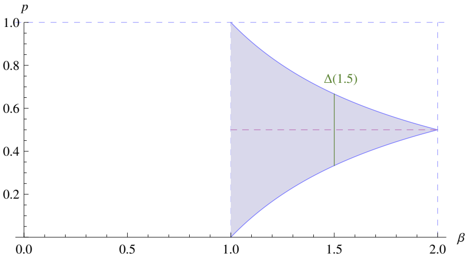
- Is there a (β, p)-shift which is SFT but **not** transitive?

Where are SFTs: **vertical** fibres

Where are SFTs: **vertical** fibres



Where are SFTs: **vertical** fibres



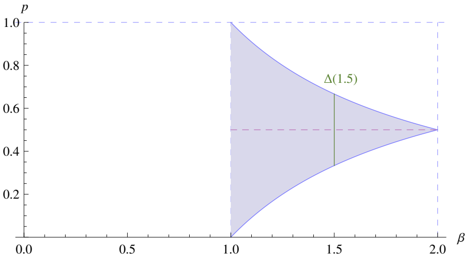
Theorem

If β is transcendental, then

$$\Delta(\beta) \cap \Delta_{\mathrm{SFT}} = \varnothing.$$

Actually, it's enough that β is not algebraic with coefficients in $\{-1, 0, 1\}$.

Where are SFTs: **vertical** fibres



Theorem

If β is transcendental, then

$$\Delta(\beta) \cap \Delta_{\mathrm{SFT}} = \varnothing.$$

Actually, it's enough that β is not algebraic with coefficients in $\{-1, 0, 1\}$.

Conjecture

If the greedy β-shift is not SFT, then $\Delta(\beta) \cap \Delta_{\mathrm{SFT}} = \varnothing$.

Where are SFTs: **vertical** fibres

Heuristics?

If β is the Golden mean, then $\Delta(\beta) \cap \Delta_{\mathrm{SFT}}$ is dense on the fibre $\Delta(\beta)$.

Where are SFTs: **vertical** fibres

Heuristics?

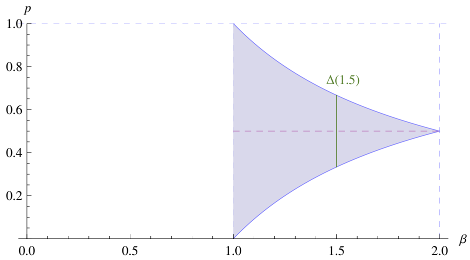
If β is the Golden mean, then $\Delta(\beta) \cap \Delta_{\mathrm{SFT}}$ is dense on the fibre $\Delta(\beta)$.

Conjecture

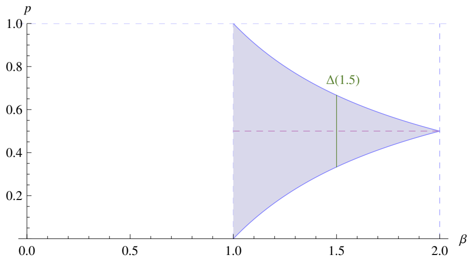
If the greedy β-shift is SFT, then $\Delta(\beta) \cap \Delta_{\mathrm{SFT}}$ is dense on the fibre $\Delta(\beta)$.

Where are SFTs: **horizontal** fibres

Where are SFTs: **horizontal** fibres



Where are SFTs: **horizontal** fibres

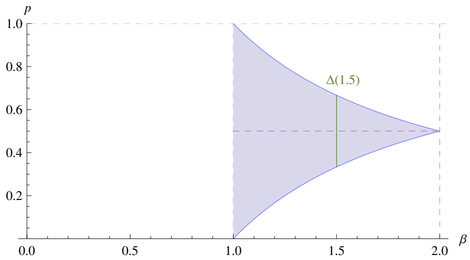


Theorem

If $p = 1/2$ and β is multinacci, i.e. $\beta^{-M} + \cdots + \beta^{-1} = 1$, then both the orbits $\{(T^{\pm}_{\beta,p})^i(p)\}_{i\in\mathbb{N}}$ are $M+1$ periodic.

In particular, (β, p)-shift is an SFT.

Where are SFTs: **horizontal** fibres



Theorem

If $p = 1/2$ and β is multinacci, i.e. $\beta^{-M} + \cdots + \beta^{-1} = 1$, then both the orbits $\{(T^{\pm}_{\beta,p})^i(p)\}_{i\in\mathbb{N}}$ are $M+1$ periodic.

In particular, (β, p)-shift is an SFT.

Conjecture

If $p = 1/2$ and the greedy β-shift is SFT, then (β, p)-shift is an SFT.

References

- P. Glendinning. Topological conjugation of Lorenz maps by β-transformations. *Math. Proc. Camb. Phil. Soc.* **107** (1990), 401-413.
- J. H. Hubbard, C. T. Sparrow, The Classification of Topologically expansive Lorenz maps. *Comm. Pure Appl. Math.* **XLIII** (1990), 431-443.
- M. R. Palmer. On the classification of measure preserving transformations of Lebesgue spaces. Ph. D. thesis (1979), University of Warwick
- N. Sidorov. Almost every number has a continuum of beta-expansions. *American Mathematical Monthly.* (2003), 110: 838-842.
- K. M. Wilkinson, Ergodic properties of a class of piecewise linear transformations. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **31** (1975) 303-328.