Bernoulli-type measures and the greedy beta-expansion

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Joint work with 李兵

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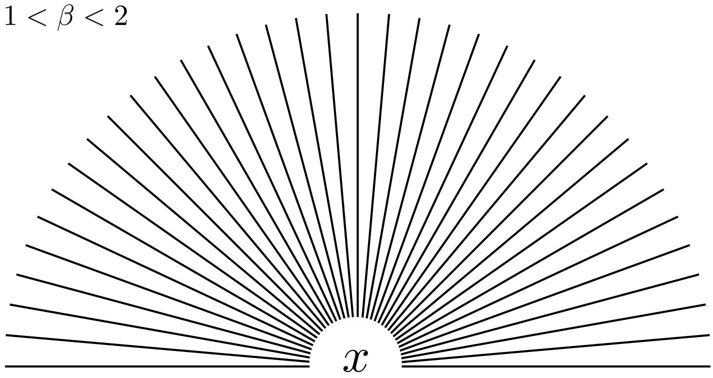

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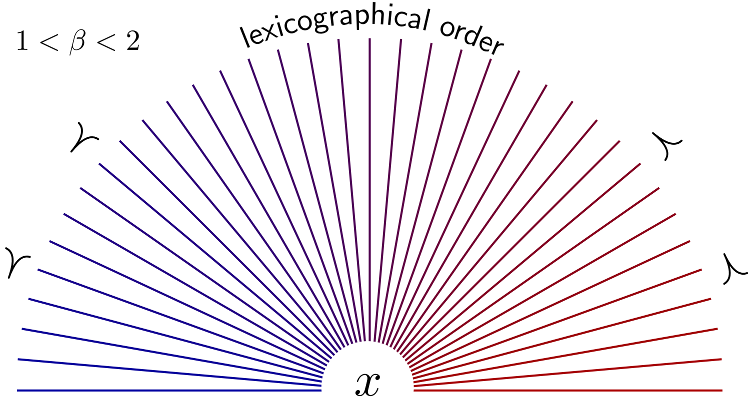

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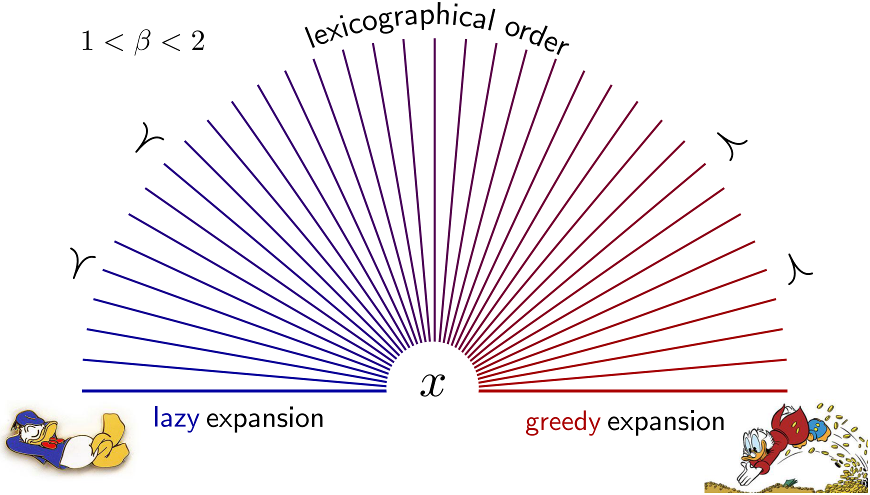

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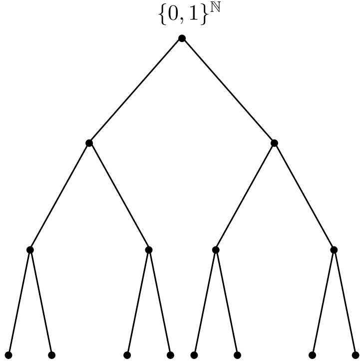

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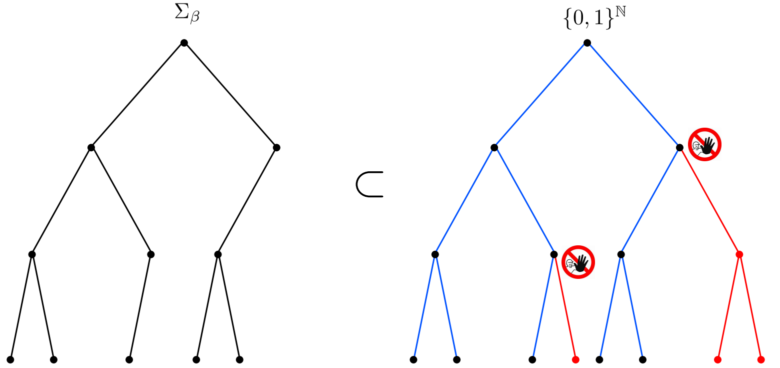

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  ...so we are done? What if $(\overline{\Sigma}_\beta, \sigma_\beta)$ is not SFT?

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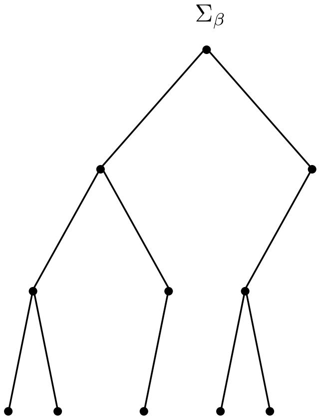

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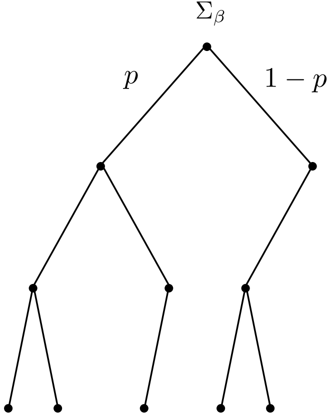

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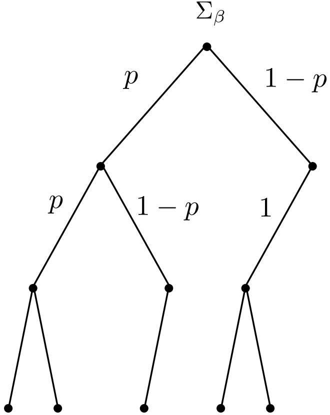

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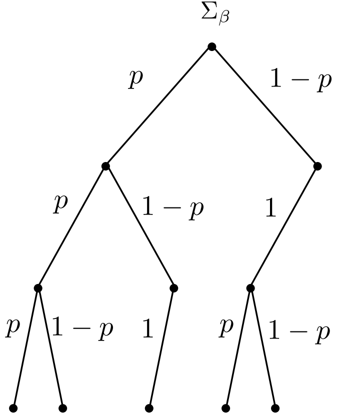

Properties of μ_p

- We are missing the precious *independence*/product structure, i.e.

$$\sigma_\beta \mu_p \neq \mu_p \quad \text{and sometimes} \quad \mu_p[ww'] \neq \mu_p[w]\mu_p[w']\dots$$

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- However, μ_p is 'quasi-invariant' in the sense that

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and also satisfies the 'ergodic property'

$$A = \sigma_\beta^{-1} A \quad \Longrightarrow \quad \mu_p(A) = 0 \text{ or } 1...$$

...but these alone do not give us strong enough ergodic theory

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Quasi-product measures

A measure μ on $\overline{\Sigma}_\beta$ is **quasi-product** (or quasi-Bernoulli), if there exists $C \geq 1$ s.t.

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Theorem (Li, S. 2013)

Let $1 < \beta < 2$ and $0 < p < 1$. Then

$$(\overline{\Sigma}_\beta, \sigma_\beta) \text{ is a subshift of finite type} \iff \mu_p \text{ is quasi-product}$$

Sketch of the proof I

- Numbers for tracking branching:

$$N_0(w) = \sharp\{1 \leq k \leq |w| : w_k = 0 \textbf{ and } w_1 \ldots w_{k-1}1 \in \Sigma_\beta^*\};$$

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- Thus

$$\mu_p[ww'] \asymp \mu_p[w]\mu_p[w'] \quad \Longleftrightarrow \quad \begin{array}{c}\text{there exists } M \ge 0 \text{ with} \\ N_0(ww') \ge N_0(w) + N_0(w') - M.\end{array}$$

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  Conjecture (A.-H. Fan 2012, personal discussion):

  (a) $\forall \beta \in (1, 2)$ $\exists$ unique $p \in (0, 1)$ s.t. $\pi_\beta \mu_p \ll \mathcal{L}$,
  (b) $\forall p \in (0, 1)$ $\exists$ unique $\beta \in (1, 2)$ s.t. $\pi_\beta \mu_p \ll \mathcal{L}$.

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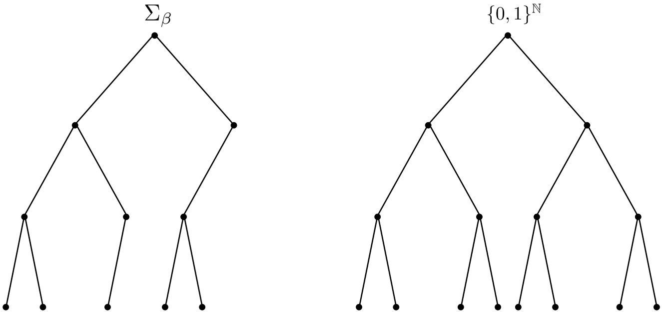

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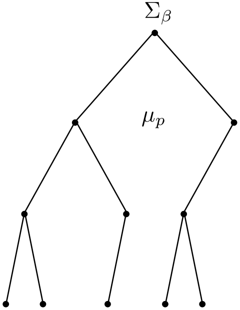

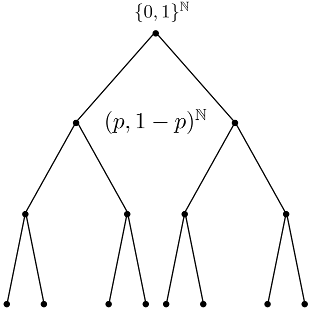

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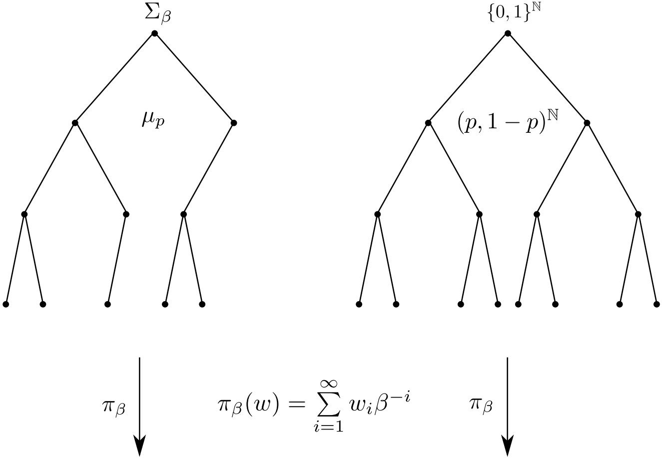

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