Brownian motion and round sets

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Leiden University, 16.5.2013

joint work with **Jonathan Fraser** and **Tuomas Orponen**

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$$|\widehat{\mu}(\xi)| \lesssim |\xi|^{-(s+\varepsilon)/2}, \xi \in \mathbb{R}^2 \quad \Longrightarrow \quad I_s(\mu) < \infty \quad \Longrightarrow \quad \dim_{\mathrm{H}} K \geq s.$$

Fourier analysis and Hausdorff dimension

This motivates...

Definition (Fourier dimension)

$$\dim_{\mathrm{F}} K := \sup\{s \leq 2 : \exists\, \mu \in \mathcal{P}(K) \text{ with } |\widehat{\mu}(\xi)| \lesssim |\xi|^{-s/2}, \xi \in \mathbb{R}^2\}$$

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Definition (Round sets)

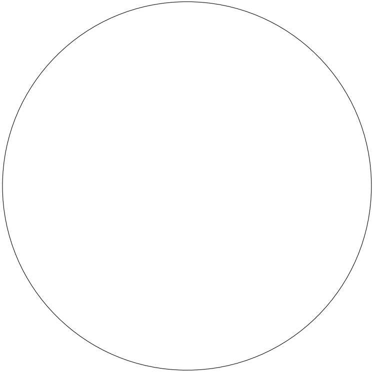
If $\dim_{\mathrm{F}} K = \dim_{\mathrm{H}} K$, we say that K is **round**.

- Round sets are also known as **Salem sets**.

Finding round sets

Finding round sets

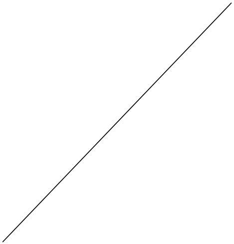
Unit circle S^1 is round...



...since $\widehat{\mathcal{H}^1 \llcorner S^1}$ decays like $|\xi|^{-1/2}$.

Finding round sets II

A line $L \subset \mathbb{R}^2$ is **not** round...



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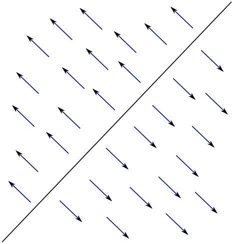

...since $\widehat{\mu} \equiv 1$ on $L^\perp$ for any $\mu \in \mathcal{P}(L)$!

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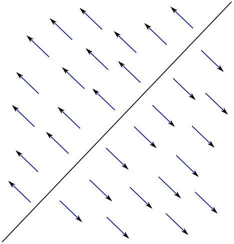

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- Punchline: $\dim_{\mathrm{H}}$ measures size, but $\dim_{\mathrm{F}}$ also contains information on **curvature**.

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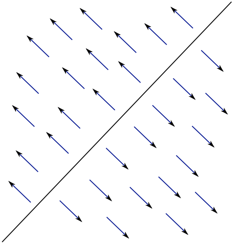

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'Non-trivial' round sets are hard to construct deterministically; but

- there are some examples by Kahane and Kaufman; in particular some sets arising from **Diophantine approximation** are round.

Finding round sets III

Many **random sets** are round...

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Many **random sets** are round... first such constructions are due to Salem, but the following key result is by Kahane.

Theorem (Kahane 1986)

Let $\omega\colon [0,\infty) \to \mathbb{R}$ *be* 1*-dimensional Brownian motion, and let* $K \subset [0,\infty)$ *be compact. Then the **image*** $\omega(K) \subset \mathbb{R}$ *is a.s. round, with*

$$\dim_{\mathrm{F}} \omega(K) = \dim_{\mathrm{H}} \omega(K) = \min\{1, 2\dim_{\mathrm{H}} K\}.$$

Analogous result also holds for fractional Brownian motion.

Finding round sets IV

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Kahane writes (1993):

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Kahane writes (1993):

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Later (2006), Shieh and Xiao explicitly ask:

> *"Are the graph and level sets of a stochastic process such as fractional Brownian motion Salem sets?"*

Finding round sets V

The conjecture is partially confirmed for level sets:

Theorem (Fouché and Mukeru 2013)

*Let $\omega : [0, \infty) \to \mathbb{R}$ be 1-dimensional (fractional) Brownian motion. Then for $a \in \mathbb{R}$, the **level set***

$$L_\omega(a) = \{0 \leq t \leq 1 : \omega(t) = a\}$$

is round with positive probability and

$$\dim_{\mathrm{F}} L_\omega(a) = \dim_{\mathrm{H}} L_\omega(a) = \frac{1}{2}.$$

Graphs of Brownian motion

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The Hausdorff dimension part is classical:

Theorem (Taylor 1953, Adler 1977)

Let $\omega : [0, \infty) \to \mathbb{R}$ be the 1-dimensional fractional Brownian motion with Hurst exponent $0 < H < 1$. Then, the ***graph***

$$G_\omega := \{(t, \omega(t)) : t \in [0, \infty)\} \subset \mathbb{R}^2$$

a.s. satisfies

$$\dim_{\mathrm{H}} G_\omega = 2 - H.$$

- In particular, $\dim_{\mathrm{H}} G_\omega > 1$ a.s.

Fourier dimension of graphs

We proved:

Theorem (Fraser, Orponen, S. 2013)

Let $E \subset \mathbb{R}$ be a set, and let $f\colon E \to \mathbb{R}$ be a function. Then

$$\dim_{\mathrm{F}} G_f \leq 1.$$

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Combining this with Taylor's and Adler's results answers Kahane's, Shieh's and Xiao's questions on random graphs in the negative:

Corollary

The Brownian graph G_ω is a.s. ***not*** *round/Salem.*

Proof

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- **Marstrand's projection theorem**: if a planar set K has Fourier dimension $s \in [0, 1]$, then all projections of K onto lines have Fourier dimension $\geq s$ (folklore).
- **Falconer's distance set conjecture**: if a planar set K has Fourier dimension $s > 1$, then the distance set of K, namely

  $$\Delta(K) = \{|x - y| : x, y \in K\},$$

  has positive length (P. Mattila).

Proof II

For us, the key example is

- **Marstrand's slicing theorem**:[1] if a planar set K has $\dim_{\mathrm{F}} K > 1$, then in every direction there are Leb positively many lines ℓ with

$$\dim_{\mathrm{H}}[K \cap \ell] > 0.$$

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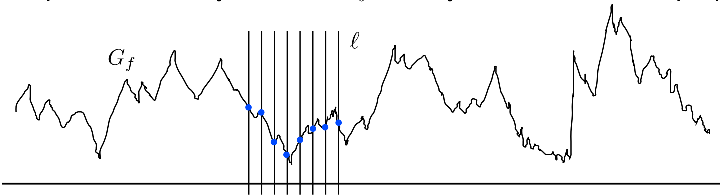
- Graphs of arbitrary functions f clearly do not have this property:



Hence, they can have Fourier dimension *at most one*!

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- **The enemy**: classical proofs do not use Fourier analysis (i.e. relation with Fourier transforms and dimension is not clear).
- **The solution**: invent a Fourier analytic proof for Marstrand's slicing theorem.

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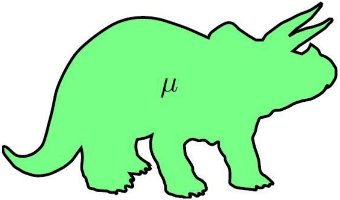
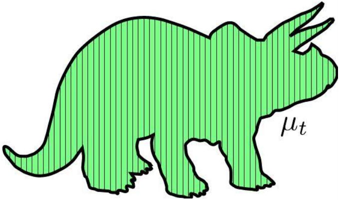
- **Slice** the measure μ with vertical lines $L_t = \{(t, y) : y \in \mathbb{R}\} \subset \mathbb{R}^2$ to obtain 'sliced measures' μ_t, supported on $K \cap L_t$.





Easy but important: $\mu_t \neq 0$ for Leb positively many t.
(this requires the decay assumption of $\widehat{\mu}$ and Plancherel's formula)

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$$\implies I_{s-1}(\mu_t) < \infty \quad \text{for Leb a.e. } t \in \mathbb{R}.$$

$$\implies \dim_{\mathrm{H}}[K \cap L_t] \geq s - 1 > 0 \quad \text{for Leb pos. many } t.$$

Q.E.D.

Open questions

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- Is Fourier dimension countably (or even finitely) stable? I.e.

$$\dim_{\mathrm{F}}\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sup_i \dim_{\mathrm{F}} A_i?$$

Further results

Not being able to solve the first open question, we considered the following variant:

Question

What is the Fourier dimension of the graph of ***(topologically) almost every*** *function* $f \in C[0,1]$*?*

- Here a set $N \subset C[0,1]$ is **topologically null** (or 'meager') if it is a countable union of nowhere dense sets in $C[0,1]$.

Further results II

We proved:

Theorem (Fraser, Orponen, S. 2013)

Almost every function $f \in C[0,1]$ *satisfies: if* $\mu \in \mathcal{P}(G_f)$, *then*

$$\limsup_{|\xi|\to\infty} |\widehat{\mu}(\xi)| \geq \frac{1}{5}.$$

In particular, $\dim_{\mathrm{F}} G_f = 0$.

- The constant $1/5$ is not sharp (optimal constant unknown).
- Hausdorff dimension $\dim_{\mathrm{H}} G_f \geq 1$ for any $f \in C[0,1]$, so our result implies that the graph of a typical function is *not* round/Salem!

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