Fourier transforms of fractal measures

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Joint work with **Thomas Jordan** (University of Bristol)

Rajchman measures

- **Fourier transform** of a Borel probability measure μ on $\mathbb{R}$ is

$$\widehat{\mu}(\xi) = \int e^{-2\pi i \xi x} \, d\mu(x), \quad \xi \in \mathbb{R}.$$

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then there are no μ supported on A with $|\widehat{\mu}(\xi)| = O(|\xi|^{-\alpha})$ if $\alpha > \frac{1}{4}$.

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- E.g. III: (Davenport-Erdős-LeVeque) If $\widehat{\mu} \to 0$ with a power decay, then μ a.e. number is normal in every base.

Kaufman measure

- Fix $N \geq 2$ and let $B_N \subset [0,1]$ be the attractor to the self-conformal iterated function system

$$\left\{ f_i : x \mapsto \frac{1}{x+i} \;\middle|\; i = 1, 2, \ldots, N \right\}$$

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Theorem (Kaufman '80 for $N \geq 3$, Queffélec-Ramaré '03 for $N = 2$)

For $N \geq 2$, there exists a Rajchman measure μ supported on B_N such that $\widehat{\mu}$ has a power decay.

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$$\mu = \sum_{\mathtt{i} \sim \mathbb{P}_n} p_{\mathtt{i}} f_{\mathtt{i}}(\mu),$$

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- Here $n = n(\xi) \in \mathbb{N}$ is a generation (depends on frequency ξ in $\widehat{\mu}(\xi)$), and $\mathbb{P}_n$ is a 'big enough' distribution of words $\mathtt{i} = i_1 \dots i_n$ s.t. the lengths of the construction intervals $f_{\mathtt{i}}[0,1]$ are mutually 'comparable'.

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- This invariance gives us the identity

  $$\widehat{\mu}(\xi) = \sum_{\mathtt{i} \sim \mathbb{P}_n} p_{\mathtt{i}} \int e^{-2\pi i \xi f_{\mathtt{i}}(x)} \, d\mu(x),$$

  where **stationary phase** bounds (via $f_{\mathtt{i}}', f_{\mathtt{i}}'', \dots$) are effective

Transfer operator

- The **transfer operator** for a *potential* $\varphi : B_N \to \mathbb{R}$ is the operator $\mathcal{L}_\varphi$ defined for a continuous $g : B_N \to \mathbb{C}$ by

$$\mathcal{L}_\varphi g(x) = \sum_{i=1}^{N} e^{\varphi(f_i(x))} g(f_i(x)), \quad x \in B_N.$$

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- **Dual** operator $\mathcal{L}_\varphi^*$ on measures is defined by $\int g \, d(\mathcal{L}_\varphi^* \mu) := \int \mathcal{L}_\varphi g \, d\mu$.
- The measures of interest for us are those that are fixed points for $\mathcal{L}_\varphi^*$, that is, those μ with

$$\mu = \mathcal{L}_\varphi^* \mu$$

i.e. for all continuous $g : B_N \to \mathbb{C}$ we have

$$\int g \, d\mu = \int \mathcal{L}_\varphi g \, d\mu.$$

Transfer operator, examples

- If μ is a Bernoulli on B_N with prob. vector $(p_1, \dots, p_N)$, then

$$\mu = \sum_{i=1}^{N} p_i f_i(\mu).$$

Thus $\mu = \mathcal{L}_\varphi^* \mu$ with the potential $\varphi(x) = \log p_{i_1(x)}$, where $i_1(x) = 1, 2, \dots, N$ is a digit with $x \in f_{i_1(x)}[0, 1]$.

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- Let $s = s_N = \dim_{\mathrm{H}} B_N$. A measure μ on B_N satisfying

$$\mu(fA) = \int_A |f'|^s \, d\mu, \quad A \subset B_N,$$

  where $f(x) = 1/x \mod 1$ is the Gauss map (the inverse branches of f are precisely f_i, $i \in \mathbb{N}$), is called a **conformal measure** for B_N.

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where $f(x) = 1/x \bmod 1$ is the Gauss map (the inverse branches of f are precisely f_i, $i \in \mathbb{N}$), is called a **conformal measure** for B_N. Thus $\mu = \mathcal{L}_\varphi^* \mu$ with the potential

$$\varphi(x) = -s \log |f'(x)|.$$

It can be shown that the Hausdorff measure $0 < \mathcal{H}^s(B_N) < \infty$ and $\mathcal{H}^s|_{B_N}$ is a conformal measure on B_N.

Fourier transforms of fixed points for $\mathcal{L}_\varphi^*$

Theorem

Suppose $\varphi : B_N \to \mathbb{R}$ is a Hölder function. If $\mu = \mathcal{L}_\varphi^* \mu$ and $\dim_{\mathrm{H}} \mu > 1/2$, then $\widehat{\mu}$ has a power decay.

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- A version of this theorem is possible for Gibbs measures (properly interpreted) on the infinite IFS $\{f_i : i = 1, 2, \dots\}$ but then one has to impose a decay condition on the **variations** of the potential φ and how $-\log|f'|$ compares to φ when we approach the cusp.

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- A version of this theorem is possible for Gibbs measures (properly interpreted) on the infinite IFS $\{f_i : i = 1, 2, \dots\}$ but then one has to impose a decay condition on the **variations** of the potential φ and how $-\log|f'|$ compares to φ when we approach the cusp.
- The main tool required in the proof is a **large deviation bound** on the generic lengths of the construction intervals $f_{\mathtt{i}}[0,1]$, which does not distort the decay bounds of $\widehat{\mu}(\xi)$ too much, but still allows us to find a similar distribution $\mathbb{P}_n$ of 'good words' Kaufman constructed for some $n = n(\xi)$ for which we can apply the stationary phase.

Applying for a grant?

- What is so special about $x \mapsto \frac{1}{x+i}$?
- Why do we need $\dim_{\mathrm{H}} \mu > 1/2$?

References

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